Random fractals, phase transitions, and negative dimension spectra

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We introduce an exactly solvable model of a random fractal which, for any finite resolution of the length scale l, exhibits a negative part of the dimension spectrum $f(\alpha)$ corresponding to the strongest singularities of the probability measure with $\alpha \approx 0$. The right section of the spectrum, corresponding to the regular part of the measure, is not well defined for $l \to 0$. These two effects are related to the fact that the generalized dimensions $\tau(q)$ exhibit (1) a first-order phase transition at $q=1$ and (2) a nonexistence of the thermodynamic limit $l \to 0$, for $q<0$. We show that an appropriate description of the scaling can be obtained by considering the logarithm of the probability of picking a singularity α on the fractal. The connections to fractal aggregates are briefly discussed.

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Multifractal formalism has become a standard tool to analyze phenomena observed in fractal aggregates, turbulence, chaotic attractors, and so on [1–4]. A multifractal object is characterized by a whole range of critical exponents α, which are the singularity exponents of the probability measure on the multifractal. Behind this statement is a hypothesis of local scaling invariance in each point of the set, since α changes from point to point. Therefore, it is necessary to introduce a continuous spectrum $f(\alpha)$ interpreted as the dimensions of the subsets $S(\alpha)$ of points which have the same α. The $f(\alpha)$ spectrum is related to the scaling exponents $\tau(q)$ of the moments of the probability measure, when the length scale $l \to 0$, by means of the Legendre transformation [1,2]:

$$\tau(q)=\min_{\alpha}[\alpha q - f(\alpha)] \ . \tag{1}$$

The standard example where all the analytic calculations can be easily performed is the two-scale Cantor set, with a fractal probability measure.

However, many fractals, such as diffusion-limited aggregates (DLA's), are described by random processes. It is thus useful to introduce a simple model of random fractals to illustrate what the differences with the corresponding deterministic model are which is usually analyzed in the literature [2,4]. In particular, it is interesting to note that in DLA's, the regular part of the arrival probability density (the harmonic measure [5]) for a diffusing probe particle seems to vanish without a local scaling because of the presence of fjords between the branches of the aggregate. The corresponding right part of the $f(\alpha)$ spectrum thus seems not be well defined and this was interpreted in Ref. [6] as a "phase transition" [7], an observation that was later elaborated in [8–15]. Our purpose is to show that such a result can be understood in a random version of the paradigmatic two-scale Cantor set.

In order to construct a deterministic Cantor set, we should split the unit interval into two intervals, say, one of length l_1 and the other of length l_2, with $l_1+l_2<1$. Each of the two intervals splits again into two new daughter intervals covering a fraction l_1 and l_2 of the mother, and so on, up to obtaining a dust of points. In order to generate a fractal probability measure on the Cantor set, one can assign a fraction of the probability p_1 to one of the intervals and $p_2=1-p_1$ to the other. At the nth stage of the construction (n here is proportional to $-\ln l$), the local scaling invariance can be described by the partition function [2]

$$\Gamma_n(q,\tau)=\sum_{i=1}^{n} p_i^q \Big/ l_i^\tau \ , \tag{2}$$

where the generalized dimensions $\tau(q)$ are obtained in the thermodynamic limit $n \to \infty$ of the exponent $\tau_n(q)$, which is a solution of the equation

$$\Gamma_n(q,\tau_n)=1 \ . \tag{3}$$

For simplicity, consider a Cantor set, where each interval has the same weight $p_i=2^{-n}$, so that the partition function can be factorized as

$$\Gamma_n(q,\tau_n)=2^{-nq}\sum_{k=0}^{n}\binom{n}{k} l_1^{-k\tau_n} l_2^{-(n-k)\tau_n} = [(l_1^{-\tau_n}+l_2^{-\tau_n})/2^q]^n \ , \tag{4}$$

and at each stage of the construction $\tau(q)$ is given by the solution of the equation $l_1^{-\tau}+l_2^{-\tau}=2^q$. The spectrum $f(\alpha)$ can be computed by the Legendre transform of $\tau(q)$; for instance, using geometrical arguments [2].

Up to now, we have considered a regular fractal, in the sense that it is constructed following a specific scheme of refinement. By contrast, we want to introduce a Cantor set, where the refinement changes from level to level. To be specific, choose a point l_1 at random with uniform probability in the interval [0, 1], and another point $1-l_2$ with uniform probability in the remaining interval $[l_1,1]$.

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This defines a "regular" Cantor set by the iterations of the fragmentation process, and we give again equal weights 2^{-n} to each of the 2^n intervals obtained at the nth step. However, it is possible consider the ensemble of realizations of these Cantor sets and look to the disorder average [over the distribution of the random variables l_1 and l_2 of the partition function (2)]

$$\langle \Gamma_n(q,\tau)\rangle = \int_0^1 dl_1 \frac{1}{1-l_1}\int_0^{1-l_1} dl_2 2^{-qn}(l_1^{-\tau}+l_2^{-\tau})^n \quad (5)$$

and determine $\tau_n(q)$ [and so $f_n(\alpha)$] by solving the equation $\langle \Gamma_n(q,\tau(q))\rangle = 1$.

On the other hand, it is easy to show that the generalized dimensions obtained by the averaged partition function [Eq. (5)] are the same obtained from a spatial average of a Cantor set, where *at each step* of the fragmentation an interval of length x splits into two intervals of random length xl_1 and xl_2, where $l_{1,2}$ are random variables distributed according to the distribution previously described. This is, of course, the correct description for random fractals. It is worth noting that in an experiment (e.g., on fractal aggregates) one can often analyze only a single realization of a random fractal. However, for large n, it is expected to exhibit the average scaling by arguments borrowed from the large deviation theory. In other experiments, such as in turbulent flows, one can only observe the first level in a multiplicative process, corresponding to $n=2$, and then ensemble averages at this level [16].

The explicit calculation of $\langle \Gamma_n(q,\tau)\rangle$ is rather interesting (see a similar result in [17]). The integral (5) does diverge for $\tau_n n \geq 1$, while for $\tau_n n \leq 1$ one obtains

$$\langle \Gamma_n \rangle = 2^{-nq}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{B(-\tau_n(n-k)+1,-\tau_n k+1)}{(1-\tau_n k)}, \quad (6)$$

where $B(x,y)$ is the Euler beta function. The convergence of the integral is thus assured by the condition

$$\tau_n(q) \leq 1/n , \quad (7)$$

so that $\lim_{n\to\infty}\tau_n(q)=\tau(q)\leq 0$. For instance, at $n=1$, one has

$$2\tau_1(q) = 2-2^{-q}-\sqrt{(2^{-q}-2)^2-4(1-2^{1-q})} .$$

The corresponding $f_1(\alpha)$ is shown in Fig. 1, where one sees that $f_1(0)=-1$.

For $n>1$, the equation $\langle \Gamma_n(q,\tau)\rangle = 1$ is transcendental and $\tau_n(q)$ has no explicit form in terms of elementary functions. However, we can give an analytic estimate for $n\to\infty$, since in this limit the sum (5) can be approximated as an integral over the variable $x=k/n\in[0,1]$. Moreover, using the Stirling formula for the Euler gamma function to estimate the beta function, one has

$$\langle \Gamma_n \rangle \approx 2^{-nq}\int_0^1 dx(-\tau_n)^{1/2} \times \frac{n^{-\tau_n}}{(-\tau_n n+1)(-\tau_n xn+1)} e^{(\tau_n+1)S(x)n} , \quad (8)$$

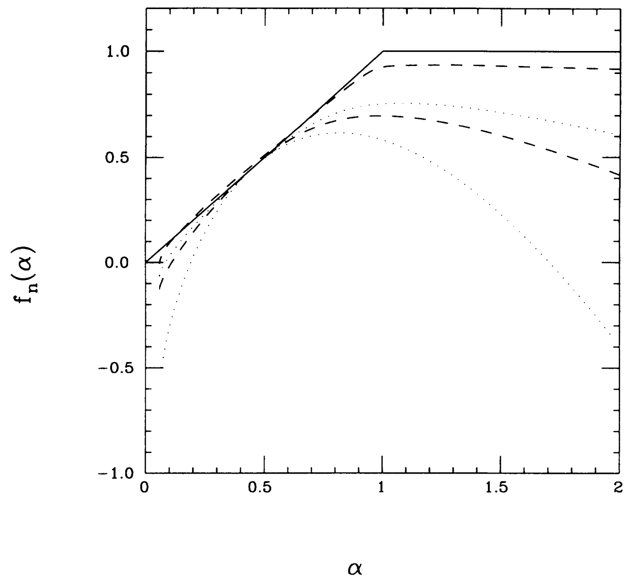

FIG. 1. $f(\alpha)$ versus α for $n=1$ (dotted line), $n=5$ (dashed line), $n=10$ (dotted line), and $n=100$ (dashed line), obtained by a Legendre transform of $\tau(q)$. The solid line is the convex envelope of the thermodynamic limit. The top of the spectrum is the fractal dimension $D_0(n)$.

with $S(x)=-x\ln x-(1-x)\ln(1-x)$. Noting that $S(x)$ is a convex non-negative function, the asymptotic behavior of the integral can be estimated by the saddle point method. We have to consider three different cases: (i) $-1\leq\tau_n\leq 0$, i.e., $0\leq q\leq 1$; (ii) $\tau_n<-1$, i.e., $q<0$ (negative moments); (iii) $\tau_n\geq 0$, i.e., $q>1$. In the first case, the maximum of the argument of the exponential in the integral (8) is reached at $x^*=\frac{1}{2}$, where $S(x)$ has its maximum, so that the saddle point method gives for $0\leq q\leq 1$

$$\langle \Gamma \rangle = (-\tau_n)^{1/2}\frac{n^{-\tau_n}}{(-\tau_n n+1)(-\tau_n n/2+1)} \times 2^{(\tau_n+1-q)n} \quad (9)$$

and

$$\tau(q)=\lim_{n\to\infty}\tau_n(q)=q-1 \quad \text{for } 0\leq q\leq 1 . \quad (10)$$

In particular, the fractal dimension of the random Cantor set is $D_0=-\tau(0)=1$, and the information dimension (fractal dimension of the set of full probability measures) $D_1=\lim_{q\to 1}d\tau/dq=1$. The corresponding part of the $f(\alpha)$ spectrum is trivial since it collapses to the point $\alpha=1$, $f=1$. However, the convergence towards the thermodynamic limit is extremely slow, as illustrated in Fig. 2, where the fractal dimension [the top of the $f(\alpha)$ spectrum] at the nth step of the construction $D_0(n)$, is plotted versus n. It is worth stressing that at $n=1$, $D_0=0.618\ldots$ (the golden mean).

In case (ii) one obtains the negative moments, which have a very interesting structure. For $\tau_n+1<0$, the argument is the exponential of the integral (8) has a max-

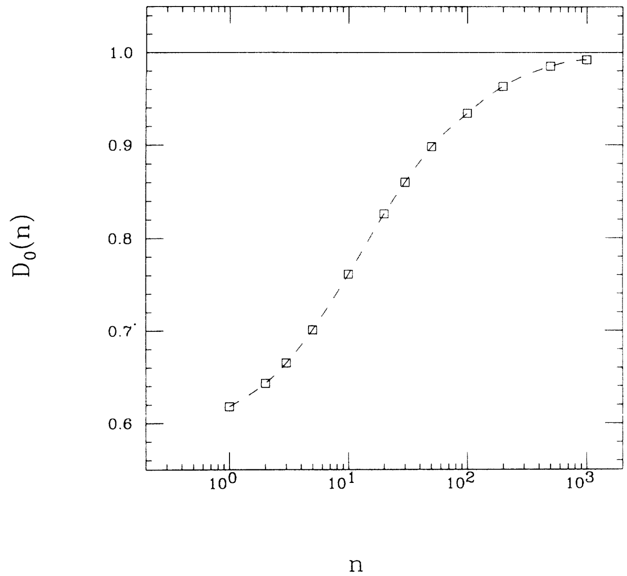

FIG. 2. Fractal dimension $D_0(n)=-\tau_n(0)$ versus n for the random Cantor set.

imum at the extrema of the integration interval, since $S(x)$ is a convex non-negative function and $S(0)=S(1)=0$. These points correspond to the terms $k=0$ and n in the original sum (6). As a consequence, the integral (8) over the continuous variable $x=k/n$ becomes a poor approximation of the sum (6), and we should directly estimate (6). The simplest approximation is to take into account only the first term ($k=0$) of (6), i.e., $2^{-nq}(1-\tau_n n)^{-1}$. Note that it is much larger than the last term ($k=n$), i.e., $2^{-nq}(1-\tau_n n)^{-2}$, when $n\to\infty$. We thus obtain

$$\tau_n(q)\approx-\frac{1}{n}(2^{n|q|}-1)-D_0(n), \quad \text{for } q\le 0 \ . \tag{11}$$

Here we have added an arbitrary constant term $\tau_n(0)=-D_0(n)$ in order to get the correct limit $q\to 0^-$ in the right hand side of (11). This shows that the thermodynamic limit of the function $\tau_n(q)$ does not exist (is infinite) for negative moments. Such a situation is not pathological, as it is numerically observed in some random fractals, such as DLA's (see Refs. [14,15]). Roughly speaking, it happens that there are regions of fractals (the fjords in DLA) which have a probability measure that is not (locally) scaling invariant because it is exponentially small in length scale [6,8,9].

We can perform the Legendre transform of (11), which gives the right part of the spectrum and reads

$$n[f_n(\alpha)-D_0(n)]\approx-\frac{\alpha}{\ln 2}(1-\ln\ln 2+\ln\alpha)-1 \quad \text{for } \alpha>1 \ . \tag{12}$$

As a consequence, in a random fractal, even if the spectrum $f(\alpha)$ itself does not exist for $\alpha>1$, one can consider the probability $P(\alpha)$ of picking at random a singularity α on the fractal

$$P(\alpha)=\frac{N_n(\alpha)}{N_n}\sim l^{-f(\alpha)+D_0(n)} \ , \tag{13}$$

where $N_n(\alpha)\sim l^{-f_n(\alpha)}$ is the number of intervals with singularity α at scale $l=2^{-n}$, and $N_n\sim l^{-D_0(n)}$ is the total number of intervals. The quantity $\ln P(\alpha)$ has a well defined thermodynamic limit, as shown in Fig. 3, where we plot $n[f(\alpha)-D_n(n)]$ versus α. On the contrary, at increasing n, the right part of $f(\alpha)$ has the typical parabolic shape which tends to the horizontal straight line $f(\alpha)=1$ for $\alpha>1$, as shown in Fig. 1. Due to the rough approximation of the sum (6) by its first term, we cannot expect a quantitative agreement between formula (6) by its first term, we cannot expect a quantitative agreement between formula (12) and the direct numerical calculation. However, it is very evident that there is a scaling of $n[f(\alpha)-D_0(n)]$ versus α instead of the usual scaling of $f(\alpha)$ versus α, according the prediction of our asymptotic estimate (12). This is probably a rather generic phenomenon, since negative moments are dominated by the most regular part of the measure, which in random fractals is expected to be absolutely continuous with respect to the Lebesgue measure.

Another peculiar characteristic of random fractals is the presence of negative $f_n(\alpha)$ which is interpreted as a negative fractal dimension in a statistical sense, meaning that it corresponds to events that occur exceptionally rarely. Although it is possible to have a negative fractal dimension or negative $f(\alpha)$ in fractals, in many cases this could be a "spurious effect" of finite volume. In fact, we can show that this is the case in our toy model. For $q>1$ one has $\tau_n(q)\ge 0$, and the convergence of the integral for $\langle \Gamma_n \rangle$ requires that $\tau_n(q)\le /n$. We should solve the equation

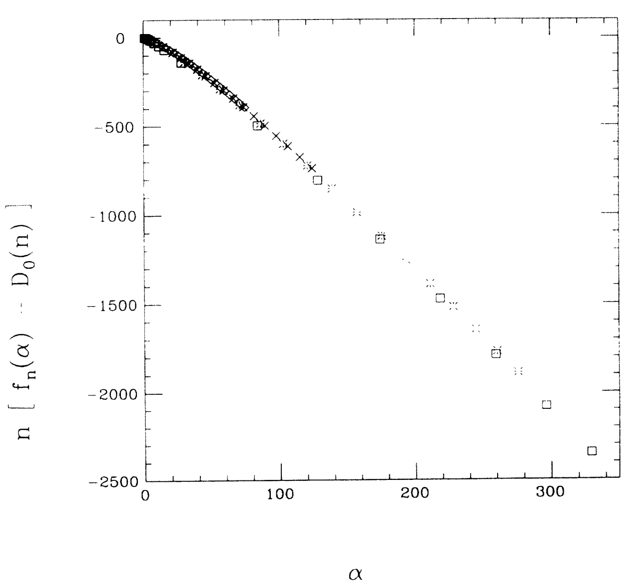

FIG. 3. $n[f_n(\alpha)-D_0(n)]$ versus α, for $\alpha\ge 1$ and $n=5$ (diamonds), 10 (crosses), 20 (stars), and 50 (squares). The results are obtained by a numerical solution of Eq. (5), where the average is over 2000 realizations.

$$\langle \Gamma_n(q,\tau_n) \rangle = \left[\frac{1}{(1-n\tau_n)} + \cdots + \frac{1}{(1-n\tau_n)^2} \right] 2^{-nq} = 1 \, . \quad (14)$$

For large q, the convergence of (5) and the convexity of τ_n assure that

$$\lim_{q \to \infty} \tau_n(q) = 1/n \quad \forall n \, . \quad (15)$$

Moreover, the dominate term of the sum (14) is its last term (a pole of the second order) so that *for all* values of n and for q that are large enough, one has

$$\tau_n(q) \approx \frac{1}{n}[1-e^{-Cnq}] \text{ with } C = \ln\sqrt{2} \, . \quad (16)$$

Its Legendre transform reads

$$n f_n(\alpha) \approx -1 + \frac{\alpha}{C} \left[1 - \ln\frac{\alpha}{C} \right] \quad (17)$$

for $\alpha \approx \alpha_{\min} = 0$, since at large q, the moments are dominated by the most singular part of the measure, which in our model are very short intervals—points—with $\alpha = 0$. Relation (17) shows that $f_n(\alpha_{\min}=0) = -1/n$ and the tail of the left part of the spectrum is negative for all finite n's. It is worth stressing that in the thermodynamic limit one has $\tau(q)=0$ for $q>1$ and a first-order phase transition [6,7] in the "potential" $\tau(q)$ at the critical point $q=1$. At this point there is a discontinuity in the first derivative of τ (a "latent heat"); that is, a jump in the $f(\alpha)$ from the point $\alpha=1$, $f=1$ [the Legendre transform of $\tau(q)=q-1$] to the point $\alpha=0$, $f=0$ [the Legendre transform of $\tau(q)=0$].

As for the negative moments, the function $\ln N_n(\alpha) = n f_n(\alpha)$ has a nontrivial thermodynamic limit, which is the correct description of the negative fractal dimensionalities $f_n(\alpha)$ of the subsets with singularity α which form the multifractal object.

In conclusion, we have discussed the simple model of a random fractal, where all the intervals have the same weight at each step of the construction. The multifractal approach cannot be applied in the standard way because of two phenomena: the nonexistence of the thermodynamic limit of the generalized dimensions $\tau(q)$ for negative q's, and a first-order phase transition at $q=1$. These behaviors are not mathematical pathologies but could be present in real random fractals, such as diffusion-limited aggregates. Moreover, we have shown that a random fractal can have a spurious negative part of the spectrum $f(\alpha)$, the fractal dimensions of the subset with the strongest singularities of the probability measure, as a consequence of finite size effects. This could be also observed in some physical phenomena such as energy dissipation in turbulent fluids, where negative $f(\alpha)$'s have been measured at rather poor resolution (large scaling parameter l) [16,18]. In this case it is not clear whether the negative spectrum disappears or not, and our results introduce a good test to decide the issue.

Our most important result is that in these cases there is a well-defined limit for the quantity related to the logarithm of the probability of finding a local scaling exponent α, $\ln P(\alpha)$ versus α. A rather similar behavior was already conjectured by Evertsz and Mandelbrot [15] in the analysis of numerical data for DLA's. It is interesting that this type of quantity has a well defined asymptotic limit not only for the negative moments but also for describing the scaling property of the most singular (and important) part of the probability measure on the random Cantor set. In fact, the thermodynamic limit of the generalized dimensions, in a statistical mechanics language a thermodynamic potential $\lim_{n\to\infty} \langle \ln Z_n \rangle / n$, where n is the particle number and Z_n the partition function, is trivially useless in both cases. On the other hand, the relevant information on the scaling properties is obtained by looking at a quantity of the type $\langle \ln Z_n \rangle$. Such as extension of multifractal formalism beyond its usual limits of applicability could be a useful tool in many different phenomena.

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[1] G. Parisi and U. Frisch in Proceedings of the International School of Physics "Enrico Fermi" Course LXXXXVIII, Varenna, 1983, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, New York, 1985), p. 84; R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984).

[2] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986); M. H. Jensen, L. P. Kadanoff, A. Libchaber, I. Procaccia, and J. Stavans, Phys. Rev. Lett. **55**, 2798 (1985).

[3] B. B. Mandelbrot, J. Fluid. Mech. **62**, 331 (1974); P. Grassberger, Phys. Lett. **97A**, 227 (1983); H. G. E. Hentschel and I. Procaccia, Physica **8D**, 435 (1983).

[4] See, e.g., G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987), and references therein.

[5] T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).

[6] T. Bohr, P. Cvitanović, and M. H. Jensen, Europhys. Lett. **6**, 445 (1988).

[7] P. Cvitanović, in *Proceedings of the Fourteenth Colloquium on Group Theoretical Methods in Physics*, edited by R. Gilmore (World Scientific, Singapore 1987); in *Non-Linear Evolution and Chaotic Phenomena*, edited by P. Zweifel, G. Gallavotti, and M. Anile (Plenum, New York, 1988).

[8] J. Lee and H. E. Stanley, Phys. Rev. Lett. **61**, 2945 (1988).

[9] R. Blumenfeld and A. Aharony, Phys. Rev. Lett. **62**, 2977 (1989); B. Fourcade and A. M. Tremblay, *ibid.* **64**, 1842 (1990).

[10] A. B. Harris and M. Cohen, Phys. Rev. A **41**, 971 (1990).

[11] B. B. Mandelbrot and T. Vicsek, J. Phys. A **20**, L377

(1989).

[12] S. Schwarzer, J. Lee, A. Bunde, S. Havlin, H. E. Roman, and H. E. Stanley, Phys. Rev. Lett. **65**, 603 (1990); J. Lee, S. Havlin, H. E. Stanley, and J. E. Kiefer, Phys. Rev. A **42**, 4.33 (1990).

[13] B. B. Mandelbrot, Physica **168A**, 96 (1990); B. B. Mandelbrot, C. J. G. Evertsz, and Y. Hayakawa, Phys. Rev. A **42**, 4528 (1990).

[14] M. Marsili and L. Pietronero, Physica **175A**, 9 (1991); **3A** (1991).

[15] C. Evertsz and B. B. Mandelbrot, Physica **185A**, 77 (1992).

[16] A. B. Chabra and K. R. Screenivasan, Phys. Rev. A **43**, 1114 (1991); Phys. Rev. Lett. **68**, 2762 (1992).

[17] G. Huber and Pl. Alstroem, J. Phys. A **24**, L1105 (1991).

[18] W. V. d. Water, J. Herweijer, and J. Molenaar (unpublished).